Announcements

1) HW #4 due Tuesday
next week

Recall: $\mathcal{L}(f')(s)$

$$= s\,\mathcal{L}(f)(s) - f(0)$$

provided $\displaystyle\lim_{x\to\infty} \frac{f(x)}{e^{xs}} = 0$!

For which functions f is this true?

Exponential Order

(Big O of an exponential function)

A function f is of exponential order α if $\exists$ $M > 0$ and $T \in \mathbb{R}$ (most of the time, $T \geq 0$) such that

$$\boxed{|f(t)| \leq M e^{\alpha t}}$$

for all $t \geq T$

(f is $O(e^{\alpha t})$)

If f is of exponential order s,

$$\left| \frac{f(x)}{e^{sx}} \right| \leq \frac{Me^{sx}}{e^{sx}} = M$$

for all $x \geq T$.

If f is of exponential order $s-1$,

$$\left| \frac{f(x)}{e^{sx}} \right| \leq \frac{Me^{(s-1)x}}{e^{sx}} = \frac{M}{e^{x}}$$

$$\to 0 \text{ as } x \to \infty$$

In general, if f is of exponential order α where $\alpha < s$,

$$\boxed{\lim_{x \to \infty} \frac{f(x)}{e^{sx}} = 0},$$

so the formula for $\mathcal{L}(f')$ holds for all such functions:

includes any polynomial, any bounded function ($\sin(x)$, $\cos(x)$, $\arctan(x)$, etc), logarithms, rational functions, ...

Back to Brine Problem

$$\frac{dx}{dt} = 6h(t) - \frac{3x(t)}{500}$$

$$h(t) = \begin{cases} .2, & 0 \leq t < 10 \\ .4, & t \geq 10 \end{cases}$$

Apply Laplace transform to both sides:

$$\mathcal{L}\left(\frac{dx}{dt}\right)(s) = \mathcal{L}\left(6h(t) - \frac{3x(t)}{500}\right)(s)$$

Using linearity of the Laplace transform, the right-hand side of the equality becomes

$$6\,\mathcal{L}(h(t))(s) - \frac{3}{500}\,\mathcal{L}(x(t))(s)$$

drop the t's to get

$$6\,\mathcal{L}(h)(s) - \frac{3}{500}\,\mathcal{L}(x)(s).$$

On the left-hand side of the equality, use the derivative rule.

$$\mathcal{L}(x')(s) = s\,\mathcal{L}(x)(s) - x(0)$$

$x(0)$ = amount of salt in the tank when $t=0$,

= 30 kg, so

$$\mathcal{L}(x')(s) = s\,\mathcal{L}(x)(s) - 30.$$

Equating both sides,

$$s\mathcal{L}(x)(s)-30 = 6\mathcal{L}(h)(s) - \frac{3\mathcal{L}(x)(s)}{500}.$$

Solve for $\mathcal{L}(x)(s)$:

$$s\mathcal{L}(x)(s) + \frac{3\mathcal{L}(x)(s)}{500} = 6\mathcal{L}(h)(s)+30$$

$$(500s+3)\mathcal{L}(x)(s) = 3000\mathcal{L}(h)(s)+15{,}000$$

Divide by $500s+3$

$$\boxed{\mathcal{L}(x)(s) = \frac{3{,}000\,\mathcal{L}(h)(s) + 15{,}000}{500s+3}}$$

Need to compute

$$\mathcal{L}(h)(s) = \int_0^\infty h(t)e^{-st}\,dt$$

$$= \int_0^{10} .2e^{-st}\,dt$$

$$+ \int_{10}^\infty .4e^{-st}\,dt$$

$$\int_0^{10} .2e^{-st}\,dt$$

$$= \frac{.2e^{-st}}{-s}\Big|_0^{10} = \frac{.2e^{-10s}}{-s} + \frac{.2}{s}$$

$$\int_{10}^{\infty} .4e^{-st}\,dt = .4\lim_{x\to\infty}\int_{10}^{x} e^{-st}\,dt$$

$$= .4\lim_{x\to\infty} \frac{e^{-st}}{-s}\Big|_{10}^{x}$$

$$= .4\lim_{x\to\infty}\left(\frac{e^{-10s}}{s} - \frac{e^{-sx}}{s}\right)$$

If $s>0$,

$$\lim_{x\to\infty} \frac{e^{-sx}}{s} = 0$$ and

if $s \leq 0$, the Laplace transform does not exist. So we have

$$\mathcal{L}(h)(s) = \frac{2}{s}\left(1 - e^{-10s} + 2e^{-10s}\right)$$

$$= \boxed{\frac{2}{s}\left(1 + e^{-10s}\right)}$$

Putting all this together,

$$\mathcal{L}(x)(s) =$$

$$\frac{3000\left(\frac{.2}{s}\left(1+e^{-10s}\right)\right)+15000}{500s+3}$$

$$= \boxed{\frac{600}{s(500s+3)} + \frac{600e^{-10s}}{s(500s+3)} + \frac{15000}{500s+3}}$$

$s>0$, so this function is continuous!

We can separate a bit using partial fractions:

$$\frac{1}{s(500s+3)} = \frac{A}{s} + \frac{B}{500s+3}$$

$$1 = A(500s+3) + Bs$$

$\underline{s=0}$ $\qquad\qquad$ $\underline{s = -\frac{3}{500}}$

$1 = 3A$ $\qquad\qquad$ $1 = -\frac{3}{500}B$

$A = 1/3$ $\qquad\qquad$ $B = -\frac{500}{3}$

Rewrite using partial fractions:

$$\boxed{\begin{aligned}&\mathcal{L}(x)(s) = \\ &600\left(\frac{1}{3s} - \frac{500}{3(500s+3)}\right) + \\ &600\left(\frac{e^{-10s}}{3s} - \frac{500e^{-10s}}{3(500s+3)}\right) \\ &+ \frac{15000}{500s+3}\end{aligned}}$$

How do we get back to $x(t)$?

Inverse Laplace Transforms

(Section 7.4)

A way back to your original function from its Laplace Transform!

Observation: The Laplace Transform, restricted to functions of exponential order that are continuous is one-to-one (up to a set of Lebesgue measure zero)

There will be an inverse for the transform!